# Poles of Eisenstein Series

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# Expectations

**Thesis:** Several computations of constant terms of interesting functions to investigate their poles.

**Talk**:

- Explain what an interesting function is.
- Explain what a constant term is.
- Explain how constant terms are related to the poles of the functions.
- Explain how L-functions arise in constant terms of Eisenstein series.

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**Talk** [illegible]

- Explain what an interesting function is.
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- Explain how L-functions [illegible] constant terms of Eisenstein series.

**Talk**: An example of part of the theory.

# Adeles

Everything is over the field $\mathbb{Q}$ . $p$ is a prime. The **finite adeles** are

$$\mathbb{A}_{\mathrm{fin}} \;:=\; {\prod_p}' \mathbb{Q}_p \;:=\; \{(x_p)_p \in \prod_p \mathbb{Q}_p : x_p \in \mathbb{Z}_p \text{ all but finitely many } p\}$$

The **adeles** are

$$\mathbb{A} \;:=\; \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$$

# Algebraic Groups

$G = \mathrm{GL}_3 : \mathrm{Alg}_{\mathbb{Q}} \to \mathrm{Groups}$

$R \mapsto 3 \times 3$ invertible matrices with $R$ entries.

# Decomposing Algebraic Groups

$$G(\mathbb{A}) = P(\mathbb{A})K,$$

- $K = \mathrm{O}_3(\mathbb{R}) \times \prod_p \mathrm{GL}_3(\mathbb{Z}_p)$.
- $P$ is one of the groups

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$$

called **standard parabolics**.

# Decomposing Algebraic Groups

$$G(\mathbb{A}) = P(\mathbb{A})K,$$

- $K = \mathrm{O}_3(\mathbb{R}) \times \prod_p \mathrm{GL}_3(\mathbb{Z}_p)$.
- These ones are

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$$

called **maximal proper standard parabolics**.

# Levi and Unipotent Subgroups

Each parabolic $P$ can be written as

$$P = MU,$$

where

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} 1,$$

$$\begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} = \begin{pmatrix} * & & \\ & * & * \\ & * & * \end{pmatrix} \begin{pmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{pmatrix},$$

$$\begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} = \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix} \begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix},$$

$$\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}.$$

# Domains

For a fixed $P = MU$, consider functions on

$$U(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A})/K \to \mathbb{C},$$

Note that if $P = G$ we get functions defined on

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K \cong G(\mathbb{Z})\backslash G(\mathbb{R})/\mathrm{O}_3(\mathbb{R})$$

Compare the $\mathrm{SL}_2$ case,

$$\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})/ \mathrm{SO}_2(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H},$$

(roughly) the domain of *modular forms*.

# Interesting Functions

By analogy consider functions

$$\phi : U(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A})/K \to \mathbb{C},$$

with properties:

- Smooth
- Moderate growth
- Some invariance under $z \in \mathrm{Lie}(G(\mathbb{R}))$

$$z.\phi(g) = \frac{d}{dt}\phi(g.e^{tz})|_{t=0}.$$

Call these (roughly) **Automorphic forms** for $\mathrm{GL}_3$.

From now fix $P = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} = MU$ as above.

Easy to check

$$\mathrm{Hom}_{\mathrm{AlgGroup}}(M, \mathrm{GL}_1) \cong \mathbb{Z}^2$$

As groups. They are maps of the form $(i,j) \in \mathbb{Z}^2$

$$(i,j) : m = \begin{pmatrix} e & & \\ & a & b \\ & c & d \end{pmatrix} \mapsto e^i \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^j$$

For $\phi$ as above, $g = muk \in M(\mathbb{A})U(\mathbb{A})K = G(\mathbb{A})$ and $(s_1, s_2) \in \mathbb{C}^2$ define

$$\begin{aligned} (s_1, s_2).\phi(g) &:= \left( \prod_\nu |(1,0)(m_\nu)|_\nu^{s_1} \, |(0,1)(m_\nu)|_\nu^{s_2} \right) \phi(g) \\ &= \left( \prod_\nu |e|_\nu^{s_1} \, |\det(ad - bc)|_\nu^{s_2} \right) \phi(g) \end{aligned}$$

# Eisenstein Series

If $g \in G(\mathbb{A}), s \in \mathbb{C}^2$ then

$$E(s,\phi)(g) \quad := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} s.\phi(\gamma g),$$

is the **Eisenstein series**.

Averages $U(\mathbb{A})M(\mathbb{Q})$ invariance into $G(\mathbb{Q})$ invariance.

Has a meromorphic continuation in the $s$ variable.

At holomorphic points it is also automorphic

$$E(s,\phi) : G(\mathbb{Q})\backslash G(\mathbb{A})/K \to \mathbb{C}.$$

# Constant Term

$\phi$ automorphic form, $P = MU$ parabolic, can take the **constant term along** $P$

$$\phi_P : M(\mathbb{A}) \to \mathbb{C},$$
$$\phi_P(m) \; := \; \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \phi(um)\mathrm{d}u.$$

This is also automorphic.

# Constant Terms of Eisenstein Series

### Theorem ([MW95], I.4.10)

*An Eisenstein series defined from $P$ has a pole at $s$ if and only if its constant term along $P$ has a pole at $s$.*

This is helpful because constant terms can be simpler.

## Example

Repeat the setup for $G = \mathrm{Sp}_{2n}$ then along maximal parabolics $P$ and for nice $\phi$

$$E_P(\phi, s) = \phi_P + M(\omega, s)\phi$$

- $\omega \in G(\mathbb{Q})$ is explicit.
- $$M(w, s)(\phi)(g) \quad := \quad \int_{\left(U(\mathbb{Q}) \cap wU(\mathbb{Q})w^{-1}\right) \backslash U(\mathbb{A})} s.\phi(w^{-1}ug)\mathrm{d}u$$

  which has relations to L-functions ...

FIN

C. Moeglin and J. L. Waldspurger.
*Spectral Decomposition and Eisenstein Series: A Paraphrase of the Scriptures*.
Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1995.